Integrable mappings via rational elliptic surfaces

2004 J. Phys. A: Math. Gen. 37 2721

(http://iopscience.iop.org/0305-4470/37/7/014)

J. Phys. A: Math. Gen. **37** (2004) 2721–2730 PII: S0305-4470(04)68076-0

Integrable mappings via rational elliptic surfaces

Teruhisa Tsuda

Graduate School of Mathematical Sciences, The University of Tokyo, Komaba, Tokyo 153-8914, Japan

E-mail: tudateru@ms.u-tokyo.ac.jp

Received 21 August 2003, in final form 28 November 2003
Published 4 February 2004
Online at stacks.iop.org/JPhysA/37/2721 (DOI: 10.1088/0305-4470/37/7/014)

Abstract
We present a geometric description of the QRT map (which is an integrable mapping introduced by Quispel, Roberts and Thompson) in terms of the addition formula of a rational elliptic surface. By this formulation, we classify all the cases when the QRT map is periodic; and show that its period is 2, 3, 4, 5 or 6. A generalization of the QRT map which acts birationally on a pencil of $K3$ surfaces, or Calabi–Yau manifolds, is also presented.

PACS numbers: 02.30.−f, 02.30.Ik, 02.40.−k, 05.45.−a

1. Introduction

We consider the birational map

$$\varphi : (x, y) \mapsto (\bar{x}, \bar{y})$$

called the *QRT map*, of the form:

$$\bar{x} = \frac{f_1(y) - f_2(y)x}{f_2(y) - f_3(y)x} \qquad \bar{y} = \frac{g_1(\bar{x}) - g_2(\bar{x})y}{g_2(\bar{x}) - g_3(\bar{x})y} \tag{1.1}$$

which was introduced in [14]. Here $f_i(y)$ and $g_i(\bar{x})$, $i = 1, 2, 3$ are the quartic polynomials:

$$\boldsymbol{f} = A\boldsymbol{y} \times B\boldsymbol{y} \qquad \boldsymbol{g} = {}^tA\bar{\boldsymbol{x}} \times {}^tB\bar{\boldsymbol{x}} \tag{1.2}$$

where $\boldsymbol{f} = {}^t(f_1, f_2, f_3)$, $\boldsymbol{g} = {}^t(g_1, g_2, g_3)$, $\boldsymbol{y} = {}^t(y^2, y, 1)$ and $\bar{\boldsymbol{x}} = {}^t(\bar{x}^2, \bar{x}, 1)$. The 3×3 matrices $A = (\alpha_{ij})_{0\leqslant i,j\leqslant 2}$ and $B = (\beta_{ij})_{0\leqslant i,j\leqslant 2}$, $\alpha_{i,j}, \beta_{i,j} \in \mathbb{C}$ play roles of the parameters of the map φ. The map φ possesses a one-parameter family of invariant curves filling the plane

$${}^t\boldsymbol{x}(A + \lambda B)\boldsymbol{y} = 0 \tag{1.3}$$

where the integration constant λ is invariant on each curve. The biquadratic curve (1.3) can be parametrized in terms of elliptic functions (see, for example, [8, 15]). Thus the QRT map φ is considered as an integrable mapping.

Various properties of the QRT map φ as an integrable mapping have already been established; φ is measure-preserving and satisfies the singularity confinement criterion (which is a discrete analogue of the Painlevé property); also, many of the discrete Painlevé equations are regarded as non-autonomous variations of the QRT map (see [4, 11, 13, 15, 16]).

In this paper, we present a geometric description of the QRT map φ. Since φ admits a one-parameter family of invariant elliptic curves (1.3), it is natural to regard φ as an auto-birational map on a pencil of elliptic curves, i.e. on a rational elliptic surface. We prove that the QRT map is an expression of the addition formula of a rational elliptic surface (see theorem 2.4). By using this geometric formulation, we can classify all the cases when the QRT map is periodic for any initial values. All the possibilities of the Mordell–Weil group $E(K)$ of a rational elliptic surface are determined in [12, corollary 2.1]. Especially the torsion subgroup $E_{\rm tor}(K) \subset E(K)$ is one of the following finite Abelian groups:

$$\{0\},\ \mathbb{Z}/2\mathbb{Z},\ \mathbb{Z}/3\mathbb{Z},\ \mathbb{Z}/4\mathbb{Z},\ \mathbb{Z}/5\mathbb{Z},\ \mathbb{Z}/6\mathbb{Z},\ (\mathbb{Z}/2\mathbb{Z})^2,\ (\mathbb{Z}/3\mathbb{Z})^2,\ \mathbb{Z}/4\mathbb{Z}\oplus\mathbb{Z}/2\mathbb{Z}.$$

Consequently, if the QRT map φ is periodic then its period is 2, 3, 4, 5 or 6; we also determine explicitly the relations among the parameters of φ for each case (see theorem 3.4, and also example 3.5).

A natural generalization of the QRT map which acts birationally on a pencil of $K3$ surfaces, or Calabi–Yau manifolds, is presented (see section 4); I expect that this map would play an important role in the study of higher-dimensional integrable mappings.

2. Geometry of the QRT map

In this section, we present a geometric description of the QRT map in terms of the addition formula of a rational elliptic surface. Consider a pencil of biquadratic curves

$$\{F(\boldsymbol{\alpha}; x, y) + \lambda F(\boldsymbol{\beta}; x, y) = 0\}_{\lambda\in\mathbb{P}^1} \tag{2.1}$$

where (x, y) is the inhomogeneous coordinate of $\mathbb{P}^1 \times \mathbb{P}^1$, and let

$$F(\boldsymbol{\alpha}; x, y) = \sum_{0\leqslant i,j\leqslant 2} \alpha_{ij} x^{2-i} y^{2-j}. \tag{2.2}$$

Note that the pencil (2.1) has eight base points (see remark 2.1). Let $P = (x, y)$ be a generic point in $\mathbb{P}^1 \times \mathbb{P}^1$, then we have a unique curve C of the pencil (2.1) passing through P.

Let us consider the line which is parallel to the x-axis and passes through P; and denote it by l_1. Then it is easy to see that the line l_1 intersects the biquadratic curve C at two points P and $\tilde{P} = (\bar{x}, y)$. The coordinate $\bar{x}$ can be written rationally in (x, y) as follows; since C passes both the points P and $\tilde{P}$, we have

$$F(\boldsymbol{\alpha}; x, y) + \lambda F(\boldsymbol{\beta}; x, y) = 0 \tag{2.3}$$

$$F(\boldsymbol{\alpha}; \bar{x}, y) + \lambda F(\boldsymbol{\beta}; \bar{x}, y) = 0. \tag{2.4}$$

Eliminating λ, we have

$$F(\boldsymbol{\alpha}; \bar{x}, y)F(\boldsymbol{\beta}; x, y) - F(\boldsymbol{\alpha}; x, y)F(\boldsymbol{\beta}; \bar{x}, y) = 0 \tag{2.5}$$

by using the notation given in (1.2), formula (2.5) can be written as

$$f_1(\bar{x} - x) - f_2(\bar{x}^2 - x^2) + f_3(\bar{x}^2 x - \bar{x}x^2) = 0. \tag{2.6}$$

Hence we obtain

$$\bar{x} = \frac{f_1(y) - f_2(y)x}{f_2(y) - f_3(y)x}. \tag{2.7}$$

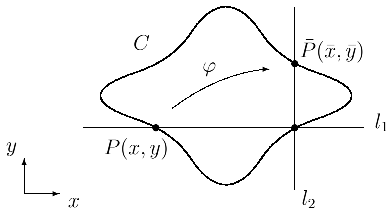

Figure 1. QRT map.

In the same way, let l_2 be the line parallel to the y-axis passing through $\tilde{P}$, then l_2 intersects C at $\tilde{P}$ and $\bar{P} = (\bar{x}, \bar{y})$, where $\bar{y}$ is given as

$$\bar{y} = \frac{g_1(\bar{x}) - g_2(\bar{x})y}{g_2(\bar{x}) - g_3(\bar{x})y}. \tag{2.8}$$

The above correspondence $P \mapsto \bar{P}$ (see figure 1) is equivalent to the QRT map (1.1).

Remark 2.1. The base points of pencil (2.1) are given as follows:

$$x = \frac{f_1(y)}{f_2(y)} = \frac{f_2(y)}{f_3(y)} \qquad \text{or} \qquad y = \frac{g_1(x)}{g_2(x)} = \frac{g_2(x)}{g_3(x)}. \tag{2.9}$$

The QRT map is indetermined at these base points in the sense of a rational map.

Remark 2.2. Viewing the above geometric construction, it is clear that the QRT map φ commutes with a projective transformation of $\mathbb{P}^1 \times \mathbb{P}^1$; we fix three points of the base points of the pencil (2.1) into (0, 0), (1, 1) and (∞, ∞). By considering a certain projective transformation of $\lambda \in \mathbb{P}^1$, we can set, without loss of generality, the 3×3 matrices $A = (\alpha_{ij})_{0\leqslant i,j\leqslant 2}$ and $B = (\beta_{ij})_{0\leqslant i,j\leqslant 2}$ as follows:

$$\begin{aligned} &\alpha_{00} = \alpha_{22} = \beta_{00} = \beta_{22} = 0 \\ &\alpha_{20} = \beta_{02} = 0 \\ &\sum_{0\leqslant i,j\leqslant 2} \alpha_{ij} = \sum_{0\leqslant i,j\leqslant 2} \beta_{ij} = 0. \end{aligned} \tag{2.10}$$

Noting that the action $A \mapsto cA$ and $B \mapsto c'B$ (c, c': constants) does not concern the expression of φ, we see that the QRT maps essentially form an eight-parameter family of integrable mappings.

Remark 2.3. The geometric construction of the QRT map has been considered in a different manner by Veselov [21]. We refer recent results of Iatrou and Roberts [7, 8], where a general class of integrable mappings of the plane has been proposed by the use of a one-parameter family of biquadratics.

From now on, we set the parameters $\alpha_{ij}, \beta_{ij} \in \mathbb{C}$ as

$$\alpha_{00} = \alpha_{22} = \beta_{00} = \beta_{22} = 0 \tag{2.11}$$

without loss of generality (see remark 2.2). Consider a biquadratic curve C_λ

$$F(\gamma; x, y) = \sum_{0\leqslant i,j\leqslant 2} \gamma_{ij} x^{2-i} y^{2-j} = 0 \tag{2.12}$$

with

$$\gamma_{ij} = \alpha_{ij} + \lambda\beta_{ij} \tag{2.13}$$

note that $\gamma_{00} = \gamma_{22} = 0$.

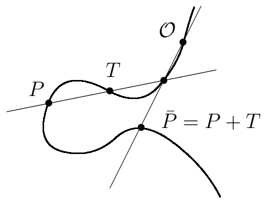

Figure 2. Addition of E_λ.

Theorem 2.4. *Let E_λ be the elliptic curve defined by the following Weierstrass equation:*

$$f_\lambda(x, y) = y^2 + a_1xy + a_3y - x^3 - a_2x^2 - a_4x - a_6 = 0 \tag{2.14}$$

where $a_6 = 0$ and $a_i = a_i(\lambda)$, $i = 1, 2, 3, 4$ are polynomials in λ of degree i given as

$$\begin{aligned} &a_1 = -\gamma_{11} \qquad a_2 = -\gamma_{01}\gamma_{21} + 2\gamma_{02}\gamma_{20} - \gamma_{10}\gamma_{12} \\ &a_3 = \gamma_{01}\gamma_{12}\gamma_{20} + \gamma_{02}\gamma_{10}\gamma_{21} - \gamma_{02}\gamma_{11}\gamma_{20} \\ &a_4 = (\gamma_{02}\gamma_{20} - \gamma_{01}\gamma_{21})(\gamma_{02}\gamma_{20} - \gamma_{10}\gamma_{12}) \end{aligned} \tag{2.15}$$

with (2.13). Then the QRT map (1.1) is equivalent to the addition formula of E_λ (see figure 2):

$$\bar{P} = P + T \qquad T = (0, 0). \tag{2.16}$$

Proof. First we consider the birational transformation $\rho : \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^2$

$$\rho : (x, y) \mapsto [X, Y, Z] = [x, y, 1]$$

where $[X, Y, Z]$ denotes the homogeneous coordinate of $\mathbb{P}^2$; then $C_\lambda{}' = \rho(C_\lambda)$ is a cubic curve in $\mathbb{P}^2$

$$\sum_{0\leqslant i,j\leqslant 2} \gamma_{ij} X^{2-i} Y^{2-j} Z^{i+j-1} = 0 \tag{2.17}$$

which passes through $T' = [1, 0, 0] = \rho(\{x = \infty\})$, $\mathcal{O} = [0, 1, 0] = \rho(\{y = \infty\})$ and $[0, 0, 1] = \rho((0, 0))$. We see that the QRT map can be described as the addition of points on the cubic curve C_λ

$$\bar{P} + \mathcal{O} = P + T'. \tag{2.18}$$

Next, we will transform the cubic $C_\lambda{}'$ into the Weierstrass normal form in a standard manner (see [20, I] and [18, III]). We start by taking the Z-axis to be tangent to $C_\lambda{}'$ at $\mathcal{O} = [0, 1, 0]$; then the tangent line intersects $C_\lambda{}'$ at another point; and take the X-axis to be tangent to $C_\lambda{}'$ at this new point. We can choose the Y-axis to be any line, other than the Z-axis, passing through $\mathcal{O} = [0, 1, 0]$. Now if we put $x = X/Z$ and $y = XY/Z^2$, then the equation for $C_\lambda{}'$ takes the form (2.14) which we call the Weierstrass equation. By a certain change of the variables fixing $\mathcal{O} = [0, 1, 0]$, preserving the Weierstrass form and taking $T' = \rho(\{x = \infty\})$ into $T = [0, 0, 1]$, we finally obtain the Weierstrass equation (2.14) with a_i given in (2.15). Summarizing above, the birational transformation $\sigma : \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^2$, which takes equation (2.12) for the curve C_λ into the Weierstrass equation (2.14), is explicitly given as follows:

$$\begin{aligned} &\sigma : (x, y) \mapsto [X, Y, Z] = [\tilde{x}, \tilde{y}, 1] \\ &\tilde{x} = \frac{(\gamma_{10}\gamma_{11}\gamma_{20} - \gamma_{01}\gamma_{20}{}^2 - \gamma_{10}{}^2\gamma_{21})y + \gamma_{20}(\gamma_{10}\gamma_{12} - \gamma_{02}\gamma_{20})}{\gamma_{10}x + \gamma_{20}} \end{aligned} \tag{2.19}$$

$$\tilde{y} = \frac{(\gamma_{01}\gamma_{20}x + \gamma_{11}\gamma_{20} - \gamma_{10}\gamma_{21})\tilde{x} + \gamma_{10}\gamma_{20}(\gamma_{10}\gamma_{12} - \gamma_{02}\gamma_{20})y + \delta}{\gamma_{10}x + \gamma_{20}} \tag{2.20}$$

where

$$\delta = \gamma_{02}\gamma_{11}\gamma_{20}{}^2 - \gamma_{01}\gamma_{12}\gamma_{20}{}^2 + \gamma_{10}{}^2\gamma_{12}\gamma_{21} - 2\gamma_{02}\gamma_{10}\gamma_{20}\gamma_{21}. \tag{2.21}$$

Since the group law of elliptic curves is unchanged under birational equivalence (see [2, section 7]), the QRT map is described in terms of the addition of E_λ as $\bar{P} = P + T$. □

Remark 2.5. The Weierstrass equation (2.14) defines an elliptic curve over the rational function field $K = \mathbb{C}(\lambda)$, i.e. an elliptic surface. Moreover, since $\mathbb{C}(x, y, \lambda) = \mathbb{C}(x, y)$, E_λ defines a rational elliptic surface. Note that E_λ realizes the full eight-parameter family of rational elliptic surfaces (see also remark 2.2).

Remark 2.6. This geometric formulation given in theorem 2.4 makes it clear that the QRT map is an autonomous limit of Sakai's elliptic Painlevé equation, which is located on the top of the all discrete and continuous Painlevé equations (see [9, 17]).

3. Torsion point of a rational elliptic surface

In this section, by using the geometric formulation of the QRT map given in section 2, we classify all the cases when the QRT map is periodic for any initial values.

Let E be an elliptic curve defined over a field K. The group of K-rational points of E is called the *Mordell–Weil group* and denoted by $E(K)$ (see [2, 18–20]). For a rational elliptic surface, the Mordell–Weil group is determined as follows:

Proposition 3.1 (see [12, corollary 2.1]). *As an abstract group, the Mordell–Weil group of a rational elliptic surface is one of the following 26 groups:*

$$\mathbb{Z}^r\ (1 \leqslant r \leqslant 8) \quad \mathbb{Z}^r \oplus \mathbb{Z}/2\mathbb{Z}\ (1 \leqslant r \leqslant 4) \quad \mathbb{Z}^r \oplus \mathbb{Z}/3\mathbb{Z}\ (1 \leqslant r \leqslant 2)$$
$$\mathbb{Z}^r \oplus (\mathbb{Z}/2\mathbb{Z})^2\ (1 \leqslant r \leqslant 2) \quad \mathbb{Z}^r \oplus \mathbb{Z}/4\mathbb{Z}\ (r = 1)$$
$$\mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \quad (\mathbb{Z}/3\mathbb{Z})^2 \quad (\mathbb{Z}/2\mathbb{Z})^2$$
$$\mathbb{Z}/6\mathbb{Z} \quad \mathbb{Z}/5\mathbb{Z} \quad \mathbb{Z}/4\mathbb{Z} \quad \mathbb{Z}/3\mathbb{Z} \quad \mathbb{Z}/2\mathbb{Z} \quad \{0\}.$$

A point T of an elliptic curve is said to have *order m* if

$$mT = \underbrace{T + T + \cdots + T}_{m} = \mathcal{O}$$

but $m'T \neq \mathcal{O}$ for all integers $1 \leqslant m' < m$. If such m exists, then T is a *point of finite order*, or a *torsion point*; otherwise it is a *point of infinite order*.

It follows from proposition 3.1 that a rational torsion point on a rational elliptic surface has order 2, 3, 4, 5 or 6. The structure of torsion points is given in the following two lemmas:

Lemma 3.2 (see [20, II]). *Let E be an elliptic curve over a field K defined by the Weierstrass equation*

$$E: \quad y^2 + a_1xy + a_3y - x^3 - a_2x^2 - a_4x - a_6 = 0.$$

(a) A point T on E has order 2 if and only if the tangent at T goes through $\mathcal{O} = [0, 1, 0]$.
(b) A point T on E has order 3 if and only if T is a point of inflection.

Lemma 3.3 (see [10, table 3])**.** *Let $E_{u,v}$ be an elliptic curve over a field K defined by the Weierstrass equation*

$$E_{u,v}: \quad y^2 + uxy + vy = x^3 + vx^2.$$

A point $T = (0, 0)$ on $E_{u,v}$ has order $n = 4, 5, 6$ if and only if $u, v \in K$ satisfy respectively

$$\begin{aligned} &(c) \quad \text{for} \quad n = 4 \quad u = 1 \\ &(d) \quad \text{for} \quad n = 5 \quad v = u - 1 \\ &(e) \quad \text{for} \quad n = 6 \quad v = -(u-1)(u-2). \end{aligned}$$

Now, we shall consider the case when the QRT map (1.1) is periodic for any initial values. Let $\Delta = \Delta(\lambda)$ be the discriminant of the Weierstrass equation (2.14):

$$\begin{aligned} &\Delta = -b_2^2 b_8 - 8b_4^3 - 27b_6^2 + 9b_2b_4b_6 \\ &b_2 = a_1^2 + 4a_2 \qquad b_4 = a_1a_3 + 2a_4 \qquad b_6 = a_3^2 + 4a_6 \\ &b_8 = a_1^2a_6 - a_1a_3a_4 + 4a_2a_6 + a_2a_3^2 - a_4^2. \end{aligned} \tag{3.1}$$

We assume that

$$\Delta \not\equiv 0. \tag{3.2}$$

This assumption ensures that the Weierstrass equation (2.14) defines a rational elliptic surface. (In the appendix, we consider the case $\Delta \equiv 0$.) Combining theorem 2.4 with the above lemmas 3.2 and 3.3, we obtain the following theorem.

Theorem 3.4. *If the QRT map φ, (1.1) with the assumption (3.2), is periodic for any initial values, then its period is 2, 3, 4, 5 or 6. Moreover, the QRT map φ has period n, $n = 2, 3, 4, 5, 6$, if and only if the parameters $A = (\alpha_{ij})_{0\leqslant i,j\leqslant 2}$, $B = (\beta_{ij})_{0\leqslant i,j\leqslant 2}$ satisfy:*

$$\begin{aligned} &(a) \quad \text{for} \quad n = 2 \qquad a_3 = 0 \\ &(b) \quad \text{for} \quad n = 3 \qquad b_8 = 0 \\ &(c) \quad \text{for} \quad n = 4 \qquad u = 1 \\ &(d) \quad \text{for} \quad n = 5 \qquad v = u - 1 \\ &(e) \quad \text{for} \quad n = 6 \qquad v = -(u-1)(u-2) \end{aligned}$$

respectively. Here $a_i = a_i(\lambda)$ are polynomials given in (2.15) with (2.13); and let

$$\begin{aligned} &b_8 = -a_1a_3a_4 + a_2a_3^2 - a_4^2 \\ &b_4 = a_1a_3 + 2a_4 \\ &u = \frac{b_4b_8}{a_3^4} \qquad v = \frac{b_8^3}{a_3^8}. \end{aligned} \tag{3.3}$$

Proof. The first statement follows from theorem 2.4 and proposition 3.1 immediately.

Take a tangent at $T = (0, 0) = [0, 0, 1]$ of the Weierstrass equation E_λ ((2.14)), and denote it by l_T. Then, the tangent line l_T is defined by the equation

$$a_4x - a_3y = 0. \tag{3.4}$$

We have the following: (a) the line l_T goes through $\mathcal{O} = [0, 1, 0]$ iff $a_3 = 0$; (b) the point T is an inflection point iff $b_8 = 0$.

Now, let us consider the change of the variables $(x, y) \mapsto (x', y')$ as

$$x = \xi^2 x' \qquad y = \xi^3 y' + \xi^2 \eta x' \tag{3.5}$$

where $\xi = a_3^3/b_8$ and $\eta = a_4/a_3$. Then E_λ (2.14) is transformed into the equation

$$y^2 + uxy + vy = x^3 + vx^2 \tag{3.6}$$

with the coefficients u and v given in (3.3). Note that transformation (3.5) fixes $T = (0, 0)$. From lemma 3.3, we have: (c) the point T is a 4-torsion point iff $u = 1$; (d) the point T is a 5-torsion point iff $v = u - 1$; (e) the point T is a 6-torsion point iff $v = -(u - 1)(u - 2)$. □

Example 3.5. We show some examples of the n-periodic QRT maps; we give below the parameters $A = (\alpha_{ij})_{0\leqslant i,j\leqslant 2}$, $B = (\beta_{ij})_{0\leqslant i,j\leqslant 2}$ and the expression of the associated QRT map $\varphi : (x, y) \mapsto (\bar{x}, \bar{y})$ for each case $n = 2, 3, 4, 5, 6$. One can easily check indeed $\varphi^n = \mathrm{id}$.

(a) $\mathbb{Z}/2\mathbb{Z}$

$$A = \begin{pmatrix} 0 & \alpha_{01} & 0 \\ \alpha_{10} & \alpha_{11} & \alpha_{12} \\ 0 & \alpha_{21} & 0 \end{pmatrix} \qquad B = \begin{pmatrix} 0 & \beta_{01} & 0 \\ \beta_{10} & \beta_{11} & \beta_{12} \\ 0 & \beta_{21} & 0 \end{pmatrix} \tag{3.7}$$

$$\begin{aligned} \bar{x} &= -\frac{p_{1221} + p_{1121}y + p_{0121}xy + p_{1021}y^2}{p_{0112}x + p_{0121}y + p_{0111}xy + p_{0110}xy^2} \\ \bar{y} &= \frac{p_{1221} - p_{1112}x - p_{0112}x^2 - p_{1012}xy}{p_{1012}x + p_{1021}y + p_{1011}xy - p_{0110}x^2y} \end{aligned} \tag{3.8}$$

where we put $p_{ijkl} = \det\begin{vmatrix} \alpha_{ij} & \alpha_{kl} \\ \beta_{ij} & \beta_{kl} \end{vmatrix}$.

(b) $\mathbb{Z}/3\mathbb{Z}$

$$A = \begin{pmatrix} 0 & p & 1 \\ 0 & r & q \\ 0 & 0 & 0 \end{pmatrix} \qquad B = \begin{pmatrix} 0 & 0 & 0 \\ 1/q & s & 0 \\ 1 & 1/p & 0 \end{pmatrix} \qquad pq \neq 0 \tag{3.9}$$

$$\bar{x} = \frac{-q(q + x + ry + pxy)}{q + pqsx + pqy + pxy} \qquad \bar{y} = \frac{q}{px}. \tag{3.10}$$

(c) $\mathbb{Z}/4\mathbb{Z}$

$$A = \begin{pmatrix} 0 & p & 0 \\ 0 & -1 & q \\ 0 & 0 & 0 \end{pmatrix} \qquad B = \begin{pmatrix} 0 & 0 & 0 \\ -p & 0 & r \\ 1 & 0 & 0 \end{pmatrix} \qquad pq \neq 0 \tag{3.11}$$

$$\bar{x} = \frac{y(q - y + pxy)}{p(-rx - y^2 + pxy^2)} \qquad \bar{y} = \frac{y(1 - px)}{px}. \tag{3.12}$$

(d) $\mathbb{Z}/5\mathbb{Z}$

$$A = \begin{pmatrix} 0 & p & 0 \\ 0 & -1 & q \\ 0 & 0 & 0 \end{pmatrix} \qquad B = \begin{pmatrix} 0 & 0 & 0 \\ -p & -pq & 0 \\ 1 & 0 & 0 \end{pmatrix} \qquad pq \neq 0 \tag{3.13}$$

$$\bar{x} = -\frac{q - y + pxy}{p(y - pqx - pxy)} \qquad \bar{y} = \frac{y - pqx - pxy}{px(1 - px)}. \tag{3.14}$$

(e) $\mathbb{Z}/6\mathbb{Z}$

$$A = \begin{pmatrix} 0 & p & 0 \\ 0 & -1 & q \\ 0 & 0 & 0 \end{pmatrix} \qquad B = \begin{pmatrix} 0 & 0 & 0 \\ -p & -pq & -pq^2 \\ 1 & 0 & 0 \end{pmatrix} \qquad pq \neq 0 \tag{3.15}$$

$$\bar{x} = \frac{y(q - y + pxy)}{p(pq^2x + pqxy - y^2 + pxy^2)} \qquad \bar{y} = \frac{y(y - 2pqx - pxy + p^2qx^2)}{px(y - pqx - pxy)}. \tag{3.16}$$

4. A birational map via a pencil of $K3$ surfaces, or Calabi–Yau manifolds

In this section we present a generalization of the QRT map which acts birationally on a pencil of $K3$ surfaces, or Calabi–Yau manifolds.

Let $I = \{m = (m_1, \ldots, m_N); m_i = 0, 1, 2\}$, and let $x = (x_1, \ldots, x_N)$ be the inhomogeneous coordinate of $(\mathbb{P}^1)^N = \underbrace{\mathbb{P}^1 \times \cdots \times \mathbb{P}^1}_{N}$. For $x = (x_1, \ldots, x_N)$ and $m \in I$, we set $x^m = x_1^{m_1} \cdots x_N^{m_N}$. Consider a pencil of polynomials in x of degree $(2, 2, \ldots, 2)$ defined as follows:

$$\{F(\boldsymbol{\alpha}^{(0)}; x) + \lambda F(\boldsymbol{\alpha}^{(1)}; x) = 0\}_{\lambda \in \mathbb{P}^1} \tag{4.1}$$

where

$$F(\boldsymbol{\alpha}^{(i)}; x) = \sum_{m \in I} \alpha_m^{(i)} x^m \qquad \alpha_m^{(i)} \in \mathbb{C} \qquad i = 0, 1. \tag{4.2}$$

For a generic point $P = x = (x_1, \ldots, x_N) \in (\mathbb{P}^1)^N$, there exists a unique element M of the pencil (4.1) which contains P. Take a line parallel to the line $\{x_i = 0; i \neq k\}$ passing through P; and denote it by l_k. The line l_k intersects M at another point; and denote this new point by $\tilde{P} = \tilde{x} = (\tilde{x}_1, \ldots, \tilde{x}_N)$. Then we obtain a birational map

$$r_k : x \mapsto \tilde{x}$$

of the form

$$\tilde{x}_i = x_i \quad (i \neq k) \qquad \tilde{x}_k = \frac{f_k - g_k x_k}{g_k - h_k x_k}. \tag{4.3}$$

Here f_k, g_k and h_k are polynomials in $(x_1, \ldots, x_{k-1}, x_{k+1}, \ldots, x_N)$ defined by

$$\begin{pmatrix} f_k \\ g_k \\ h_k \end{pmatrix} = \begin{pmatrix} A_k^{(0)} \\ B_k^{(0)} \\ C_k^{(0)} \end{pmatrix} \times \begin{pmatrix} A_k^{(1)} \\ B_k^{(1)} \\ C_k^{(1)} \end{pmatrix} \tag{4.4}$$

with

$$A_k^{(i)} = \sum_{m_k=2} \alpha_m^{(i)} x^m x_k^{-2} \qquad B_k^{(i)} = \sum_{m_k=1} \alpha_m^{(i)} x^m x_k^{-1} \qquad C_k^{(i)} = \sum_{m_k=0} \alpha_m^{(i)} x^m \tag{4.5}$$

i.e.,

$$F(\boldsymbol{\alpha}^{(i)}; x) = A_k^{(i)} x_k^2 + B_k^{(i)} x_k + C_k^{(i)}. \tag{4.6}$$

Note that $r_k^2 = \mathrm{id}$. We now define a birational map $\Phi : (\mathbb{P}^1)^N \to (\mathbb{P}^1)^N$ by

$$\Phi = r_N \circ r_{N-1} \circ \cdots \circ r_1 \tag{4.7}$$

which is regarded as a generalization of the QRT map (1.1); for $N = 2$, the map Φ is equivalent to the QRT map (see section 2).

Remark 4.1. Equation (4.2) in $(\mathbb{P}^1)^N$ generally defines

(a) an elliptic curve for $N = 2$
(b) a $K3$ surface for $N = 3$
(c) a Calabi–Yau manifold for $N \geqslant 4$.

Here we recall that an n-dimensional projective smooth algebraic variety X is said to be a $K3$ surface for $n = 2$, and a Calabi–Yau manifold for $n \geqslant 3$, if $K_X \sim 0$ and $H^i(X, \mathcal{O}_X) = 0\,(1 \leqslant i < n)$ (see for example [3]).

Remark 4.2. Some special cases of the map Φ for $N = 4$ are considered in [1], from the view point of discrete integrable systems.

Acknowledgments

The author would like to thank Professors Mitsuaki Eguchi, Masahiro Kanai, Toshiyuki Katsura, Yoichi Miyaoka, Mikio Murata, Kazuo Okamoto and Masaaki Suzuki for valuable discussions. The author also expresses his thanks to the referees who have given him useful comments about previous works on this subject.

Appendix. *n*-Periodic QRT maps with $\Delta \equiv 0$ (by Mikio Murata)

We consider the case $\Delta \equiv 0$, namely, when the Weierstrass equation (2.14) does not define any rational elliptic surface but a Hirzebruch surface ($\mathbb{P}^1$-bundle over $\mathbb{P}^1$). In this case, we can construct a periodic QRT map with an arbitrary period.

For example, let n be an arbitrary positive integer, and let

$$A = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 2\cos\frac{\pi k}{n} & \sin\frac{\pi k}{n} \\ -1 & -\sin\frac{\pi k}{n} & 0 \end{pmatrix} \qquad B = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \tag{A.1}$$

where $k \in \mathbb{Z}$ is relatively prime to n; it is easy to check that $\Delta \equiv 0$. Then, the associated QRT map $\varphi : (x, y) \mapsto (\bar{x}, \bar{y})$ is given as follows:

$$\begin{aligned} \bar{x} &= \frac{\left(2x\cos\frac{\pi k}{n} - y\right)\left(x + 2x\cos\frac{\pi k}{n} - y - 2\sin\frac{\pi k}{n}\right)}{x + y} \\ \bar{y} &= \frac{x\left(x + 2x\cos\frac{\pi k}{n} - y - 2\sin\frac{\pi k}{n}\right)}{x + y} \end{aligned} \tag{A.2}$$

which has period n, i.e. $\varphi^n = \mathrm{id}$. In fact, (A.2) can be rewritten into the map $\varphi' : (\xi, \eta) \mapsto (\bar{\xi}, \bar{\eta})$,

$$\bar{\xi} = \xi \exp\left(\frac{2\pi k\sqrt{-1}}{n}\right) \qquad \bar{\eta} = \eta \tag{A.3}$$

through the change of the variables $(x, y) \mapsto (\xi, \eta)$ defined by

$$\xi = \frac{x - \mathrm{e}^{\frac{-\pi k\sqrt{-1}}{n}} y}{\mathrm{e}^{\frac{\pi k\sqrt{-1}}{n}} y - x} \qquad \eta = \frac{x^2 - 2xy\cos\frac{\pi k}{n} + y^2 - x\sin\frac{\pi k}{n} + y\sin\frac{\pi k}{n}}{x + y}. \tag{A.4}$$

Remark A.1. Several examples of periodic maps were studied in the context of discrete integrable systems (see [6] and [5, II]).

References

[1] Capel H W and Sahadevan R 2001 *Physica* A **289** 86–106
[2] Cassels J W S 1991 *Lectures on Elliptic Curves* (Cambridge: Cambridge University Press)
[3] Cox D A and Katz S 1999 *Mirror Symmetry and Algebraic Geometry* (Providence, RI: American Mathematical Society)
[4] Grammaticos B, Nijhoff F W and Ramani A 1999 Discrete Painlevé equations *The Painlevé Property* ed R Conte (New York: Springer) pp 413–516
[5] Hirota R and Takahashi D 2003 *Discrete and Ultradiscrete Systems* (Tokyo: Kyoritsu) (Japanese)
[6] Hirota R and Yahagi H 2002 *J. Phys. Soc. Japan* **71** 2867–72
[7] Iatrou A and Roberts J A G 2001 *J. Phys. A: Math. Gen.* **34** 6617–36
[8] Iatrou A and Roberts J A G 2002 *Nonlinearity* **15** 459–89
[9] Kajiwara K, Masuda T, Noumi M, Ohta Y and Yamada Y 2003 *J. Phys. A: Math. Gen.* **36** L263–72
[10] Kubert D 1976 *Proc. London Math. Soc.* (3) **33** 193–237
[11] Nobe A 2003 *Theor. Appl. Mech. Japan* **52** 229–37
[12] Oguiso K and Shioda T 1991 *Comment. Math. Univ. St. Paul.* **40** 83–99
[13] Quispel G R W, Capel H W and Scully J 2001 *J. Phys. A: Math. Gen.* **34** 2491–503
[14] Quispel G R W, Roberts J A G and Thompson C J 1989 *Physica* D **34** 183–92
[15] Ramani A, Carstea A S, Grammaticos B and Ohta Y 2002 *Physica* A **305** 437–44
[16] Sahadevan R and Quispel G R W 1997 *Physica* A **234** 775–84
[17] Sakai H 2001 *Commun. Math. Phys.* **220** 165–229
[18] Silverman J H 1986 *The Arithmetic of Elliptic Curves* (New York: Springer)
[19] Silverman J H 1994 *Advanced Topics in the Arithmetic of Elliptic Curves* (New York: Springer)
[20] Silverman J H and Tate J 1992 *Rational Points on Elliptic Curves* (New York: Springer)
[21] Veselov A P 1991 *Russ. Math. Surveys* **46** 1–51